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A Moments Method Applied to Laplace Transform Technique for Experimental Physics*

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The Laplace transform technique is widely used in engineering. In such applications all the physical constants (parameters) of a system are usually known and its precise behavior can be calculated. However, in experimental physics, especially solid state physics, the situation can be the opposite. Measurements can describe the behavior of a system and one wishes to extract its physical parameters. We present here a mathematical method which enables one to use the Laplace transform technique for this special application. This method, which can be used for a large variety of physical phenomena, makes use of the shape of the response of a system as a function of time. We present here the mathematical and physical considerations for those cases to which the method can be applied. Several examples are worked out to show the correct way to use the technique.

INTRODUCTION

The Laplace transform technique has a large number of applications in physics and engineering [1–3]. It is used mainly for the solution of the differential equations describing the physical behavior of a system under certain (given) external conditions. The applications include, among other topics, electric network theory, heat condution problems, mechanics and hydrodynamics of continuous systems, and electric transmission lines.

The limitations of the technique are twofold. The transformability of the differential equations describing the system imposes restrictions on the complexity of problems that can be solved. The second limitation is the validity of the mathematical model as a true description of the experiment. The second limitation can be checked by making a redundant calculation of the parameters.

Three steps are involved in the actual solution of a problem. In the first, the differential equations of the problem and the initial and boundary conditions imposed externally are transformed into the "image" space, that is, the Laplace transform is taken for all the equations and functions involved. This step is usually

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not difficult because the externally enforced conditions are simple functions. Published tables of Laplace transforms [4] are quite extensive, enabling one to work out this first step easily even for complicated cases. In the second step, the equations in the image space are solved for the desired variable. In experimental physics, this will be a measureable quantity like temperature in a heat conduction problem, or voltage, or current in an electrical problem; etc. We denote by $\bar{X}(p)$ the solution in the image space. The inverse transform of the function $\bar{X}(p)$ is found in the third step, yielding $X(t)$, the solution in real space. It turns out that the last step is usually the most complex of the three. In some cases, a straightforward inverse transform can be found using published tables. However, for most cases involving real life finite size systems, the solution in the real space is given in a form of infinite series which converge under certain conditions, usually for large or small values of the time. In these cases, the form of the solution is not adequate for a complete and accurate analysis of the experimental results in order to obtain the physical parameters of the system. Large computers must be used to find the parameters through complicated least-squares fits of the experimental data to the complex form of the solution.

In this paper we present a simple moments method that replaces two complex parts in the procedure outlined above. It enables one to drop the third step completely and it does not make use of a complicated least-squares fit. The exact form of $X(t)$ is not required. The moments X_n are estimated from the mathematical model as derivatives and integrals of $\bar{X}(p)$ at $p = 0$.

These results are compared to the experimental moments, which are integrals of $X(t)\,t^n$. This comparison gives a simple set of equations for the physical parameters of the system. Usually, the number of useful equations is limited only by the accuracy of the experiment (it should be noted that the experimental moments are less sensitive to instruments (electronic) noise than any particular point or derivative of $X(t)$). Therefore, in most cases it is possible to get a redundant set of equations for the physial parameters. In this way the validity of the whole mathematical model can be checked through the moments method. The exact estimation of $X(t)$ is not needed for finding the moments from the model. However, there are some limitations on the existence of X_n. In the subsequent sections we will describe the practical and correct way to find when the moments X_n exist. Several worked examples are included.

1. The Moments Method

The Laplace transform $\bar{X}(p)$ of a function $X(t)$ is defined as

$$\bar{X}(p) = \int_0^\infty e^{-pt} X(t)\, dt, \qquad \text{Real } p \geqslant C. \tag{1}$$

In this definition we assume that $X(t)$ is defined for $t > 0$ and that the integral appearing in (1) exists for Real $p \geqslant C$, where $C \geqslant 0$. $\bar{X}(p)$ is called the image function while $X(t)$ is the original function. Usually this is the time, but Laplace transforms can be defined for any variable. In this paper we shall assume that t means time and $X(t)$ is also a function of other variables such as distance, angles, initial values of the system, etc., and so, therefore, is $\bar{X}(p)$. In our treatment, $\bar{X}(p)$ is the image space solution of a measurable quantity $X(t)$ in a physical experiment. The outline of the calculation of $\bar{X}(p)$ in the image space was given in the Introduction and we will confine our work to the information that one can obtain from $\bar{X}(p)$ without using the complete form of $X(t)$.

We define the moments X_n for any integers n as

$$X_n = \int_0^\infty X(t)\, t^n\, dt, \qquad -\infty < n < \infty. \tag{2}$$

We assume that these moments X_n exist, at least for some values of n. We now apply the general rules of the Laplace transform ([2, pp. 255–257] or [3, pp. 26–27]). If $\bar{Y}(P_0)$ exists and $p \to p_0 + 0$ through real values, then

$$\lim_{p \to p_0 + 0} \bar{Y}(p) = \bar{Y}(p_0). \tag{3}$$

In particular, for $p_0 = 0^+$, and assuming continuity at the origin:

$$\lim_{p \to 0} \bar{Y}(p) = \int_0^\infty Y(t)\, dt. \tag{4}$$

Applying this result to $Y(t) = X(t)\, t^n$ and $Y(t) = X(t)\, t^{-n}$ for $n \geqslant 0$ we obtain

$$X_n = \lim_{p \to 0} (-1)^n \left[\frac{d^n \bar{X}(p)}{dp^n}\right], \qquad n \geqslant 0, \tag{5}$$

$$X_{-n} = \lim_{p \to 0} \left[\int_p^\infty \cdots \int_p^\infty \bar{X}(p)\, dp\right] \qquad (n\text{-multiple integral}) \quad n \geqslant 0. \tag{6}$$

We have thus completed the mathematical formulation of the moments method. We note that the evaluation of the moments X_n using (5) and (6) is usually an easy calculation, especially using (5). However, sometimes the computation of the multiple integral used in (6) is not straightforward. In these cases, it is sometimes easier to calculate the moments numerically using modern numerical inverse transformation techniques for $\bar{X}(p)$. We can also add that (4) can be applied to a wide variety of integrals and derivatives of $X(t)$ if they are defined, and if the corresponding Laplace transforms exist. We quote two simple examples:

$$Y(t) = \frac{dX(t)}{dt} \quad \text{and} \quad Y(t) = \int_0^t X(t)\, dt,$$

$$\int_0^\infty \left(\frac{dX(t)}{dt}\right) dt = \lim_{p\to 0} p\bar{X}(p) - X(0^+); \tag{7}$$

$$\int_0^\infty \left(\int_0^t X(T)\, dT\right) dt = \lim_{p\to 0} \frac{1}{p} \bar{X}(p). \tag{8}$$

In particular,

$$\lim_{p\to 0} p\bar{X}(p) = \lim_{t\to\infty} X(t). \tag{9}$$

It is also shown [3, p. 144; 2, p. 255] that if $\lim_{t\to 0} X(t)$ exists,

$$\lim_{p\to\infty} pX(p) = \lim_{t\to 0} X(t). \tag{10}$$

In an experiment we measure $X(t)$, and therefore the experimental moments X_n can be computed from (2). Assuming that the moments exist mathematically, they can be compared directly with the simple limiting values obtained in (5) and (6). This comparison will give a simple set of equations for the physical parameters of the system. Sometimes, an electronic measuring system does not yield $X(t)$ but rather $dX(t)/dt$ and/or $\int_0^t X(t)\, dt$. The first moments of those functions have been exhibited in (7) and (8), respectively. Extension of our method and variations can be found using any table of Laplace operations [3, p. 209; 2, p. 257]. Another extension is the use of $p_0 \neq 0$ in (3). It will be dealt with in Section 3.1.

There are two problems to be solved before the algebraic procedure outlined above can be utilized. These are:

(1) the proof that the moments do exist mathematically without having to find the exact form of $X(t)$,

(2) practical ways of finding the moments.

We shall give here mathematical restrictions on $\bar{X}(p)$, and practical and physical considerations when, and in which way, the above-mentioned problems can be overcome.

2. Mathematical Considerations

We can overcome the two problems mentioned above by finding the limiting behavior of $X(t)$ as $t \to 0$ and/or $t \to \infty$ without the necessity of finding the complete solution. This behavior will permit us to evaluate the moments and to probe their existence.

2.1. *Estimates for the Behavior of $X(t)$ for $t \to \infty$.*

We have two main approaches for estimating $X(t)$ at $t \to \infty$ without a complete solution: (1) Series expansion of $\bar{X}(p)$; (2) The inversion theorem. If $\bar{X}(p)$ can be expanded in a neighborhood of α_0 in an absolutely convergent power series with arbitrary exponents

$$\bar{X}(p) = \sum_{n=0}^{\infty} a_n(p - \alpha_0)^{\lambda_n}, \qquad -N < \lambda_0 < \lambda_1 < \cdots < \infty, \tag{11}$$

then $X(t)$ has the asymptotic expansion for $t \to \infty$,

$$X(t) \approx e^{\alpha_0 t} \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(-\lambda_n)} t^{-\lambda_n - 1}. \tag{12}$$

Nonnegative integral exponents $\lambda_n \geqslant 0$ do not contribute to $X(t)$ $[1/\Gamma(-\lambda_n) = 0]$. Usually (12) is a complicated expansion. We exhibit it because there are cases where the inversion theorem (see below) cannot be applied and we must use this expansion to examine the leading term in $X(t)$ as $t \to \infty$. The simpler approach is to use the generally accepted way of finding $X(t)$ analytically from $\bar{X}(p)$: the inversion theorem [2, Sections 29–31]. If $X(t)$ has a continuous derivative and $|X(t)| < Ke^{Ct}$, where K and C are positive constants, then

$$X(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{pt} \bar{X}(p)\, dp, \qquad \text{where} \quad \gamma > C, \quad t > 0. \tag{13}$$

The line integral is usually evaluated by transforming it into a closed contour and applying the calculus of residues. If $\bar{X}(p)$ is a single-valued function of p we complete the contour by including the line Real $p = \gamma$ and a large circle of radius R in the half-plane, not passing through any pole of the integrand. The contour is shown in Fig. 1. The integral over this circle vanishes in the limit $R \to \infty$ if

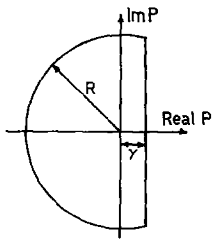

FIG. 1. The closed contour used for the application of the inversion theorem for a "regular" function $\bar{X}(p)$.

$|\bar{X}(p)| < AR^{-K}$ with $R > R_0$, $K > 0$, where R_0, K, and C are constants (Jordan's lemma [2, Section 31]). Thus, in this limit and using Cauchy's integral formula, $X(t)$ is the sum of the residues at the poles of the integrand within the contour.

A function $\bar{X}(p)$ that can be integrated using the closed contour of Fig. 1 will be called "regular."

We shall now explain the implications encountered by the mathematical restrictions on a "regular" $\bar{X}(p)$. We first look at $X(t)$. Mathematically the step and δ functions are not included in this treatment. Nevertheless, if a formal treatment of them leads to a result capable of physical interpretation, then in practical cases the result may be accepted as correct [5, pp. 65]. This concept, which is generally accepted by mathematicians [2, Section 106; 3, Section 13], removes any real a priori limitation on $X(t)$ encountered in physical applications. It has also a *rigorous* basis in distribution theory.

The limitations imposed by Jordan's lemma can be checked directly on $\bar{X}(p)$. It turns out that for finite geometry problems [1–3] this restriction is of no importance whatsoever. Only for some infinite geometries are the requirements of Jordan's lemma not fulfilled, usually when the term $e^{-\alpha p}$ ($\alpha > 0$) appears as a multiplier. This point will be clarified in Section 3.1.

The last mathematical restriction on $\bar{X}(p)$ is that it has to be single-valued. From [5, p. 535], the only elementary functions that are not single-valued are those including values of p for which a quantity raised to a nonintegral power vanishes or for which quantity whose logarithm is taken vanishes or becomes infinite. The quantity most often encountered in Laplace transform solutions $\bar{X}(p)$ is $q = (ap^2 + bp + c)^{1/2}$ where a, b, c are positive constants. However, for most finite geometry cases, $\bar{X}(q)$ is a single-valued function of p. To show this, one must check to see if $\bar{X}(q)$ is an even function of q.

We proceed now to examine $X(t)$ obtained by the inversion theorem using the contour of Fig. 1. Let p_n be the nth pole of $\bar{X}(p)$. We have three possibilities: Real $p_n > 0$, Real $p_n = 0$, and Real $p_n < 0$. We will not consider the first one, because the corresponding $X(t)$ describes an unstable system having an exponentially increasing response. (Real p_n appears in the exponent.) Real $p_n = 0$ means a purely oscillatory behavior as $t \to \infty$. We will elaborate on this case in Section 3.1. The third and most frequent possibility involves an exponential decreasing factor, so that all the moments X_n, $n \geqslant 0$, exist mathematically.

2.2. *Estimates for the Behavior of $X(t)$ for $t \to 0$*

If $\bar{X}(p)$ can be expanded in the absolutely convergent series for $|p| > R$ of the form

$$\bar{X}(p) = \sum_{n=0}^{\infty} \frac{a_n}{p^{\lambda_n}}, \qquad 0 < \lambda_0 < \lambda_1 \cdots \to \infty, \tag{14}$$

then we can express $X(t)$ in the converging series:

$$X(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\lambda_n - 1}}{\Gamma(\lambda_n)}. \tag{15}$$

This theorem is extended rigorously to transforms that can be expanded in series of negative exponentials of p, $p^{1/2}$, $(ap^2 + bp + c)^{1/2}$ $(a, b, c \geqslant 0)$ which we encounter practically. The expansions obtained can be transformed term by term to obtain $X(t)$. We usually need only the leading term as $t = 0$.

3. Physical and Practical Considerations

Let $X(p)$ be the Laplace transform solution of the mathematical model describing the experimental setup. There is a good chance that the same problem was solved sometime in the past and appears in the vast literature on Laplace transforms, For example, a large number of problems in conduction of heat, hydrodynamics, diffusion, mechanical systems, electric circuits, and transmission lines are solved in Refs. [1–3], usually in a form of infinite series for $X(t)$. We estimate the existence of the moments by looking on the leading term in $X(t)$ as $t \to 0$ and $t \to \infty$. The moments X_n are found using (5) and (6) directly on the function $\bar{X}(p)$. These moments will be used together with the experimental moments found using (2) to find the physical parameters of the system.

3.1. *Considerations for the Moments X_n, $n \geqslant 0$.*

In the general case where $X(t)$ is not known, we will use the guidelines given in Section 2 to assure the existence of the moments. First we check if $\bar{X}(p)$ is a single-valued function of p and bounded as $p \to \infty$ in the left half-plane, i.e., a "regularity" check.

If $\bar{X}(p)$ is a multivalued function, we can use the expansion shown in (11) and/or a Table of Transforms for the sole purpose of finding the leading term as $t \to \infty$. If $\bar{X}(p)$ is unbounded in the left half-plane, there is no general way to find the leading terms. However, if

$$\bar{X}(p) = e^{-ap}\bar{f}(p), \tag{16}$$

where $\bar{f}(p)$ is a "regular" function of p and $f(t)$ is its inverse transform, then

$$X(t) = H(t - a(f(t - a), \tag{17}$$

where

$$H(t-a) = \begin{cases} 0, & t < a, \\ 1, & t > a. \end{cases}$$

In this case, we can treat $\bar{X}(p)$ as any "regular" function.

All the "regular" functions $\bar{X}(p)$ can be inverted using the inversion theorem, as explained in Section 2.1. As we omit systems with exponentially increasing response, we have two types of behavior as $t \to \infty$. $X(t)$ has purely oscillatory terms or it has exponentially decreasing factors in all the terms and the system is in equilibrium as $t \to \infty$ (see Section 2.1). First we shall treat the second type. This behavior is most frequent in finite-sized daped systems if the input is of finite duration or constant. Several examples are electric circuits and transmission lines where resistance and leakage are not neglected, heat conduction samples, damped mechanical and electromagnetic wave systems, hydrodynamic problems in viscous liquids, etc. We will find the equilibrium value x of $X(t)$ as $t \to \infty$ using (9). If $x \neq 0$ we will use the function

$$X^*(t) = X(t) - x \tag{18}$$

instead of $X(t)$. $\bar{X}^*(t)$ has the Laplace transform

$$\bar{X}^*(p) = \bar{X}(p) - (x/p) \tag{19}$$

and has the limiting value $X^* = 0$ for $t \to \infty$. In all the physical systems mentioned, we know that for either $\bar{X}(p)$ or $\bar{X}^*(p)$ all the moments X_n or X_n^*, $n \geqslant 0$, exist mathematically, and the leading term as $t \to \infty$ has a negative exponent. (The leading term is $f(t)\, e^{-at}$, where $f(t)$ is a polynomial or an oscillating function. For very large values of t, the envelope of $X(t)$ behaves like an exponential.)

The steady state oscillatory case does not possess the moments as defined in (2). Let us define the pseudomoments $X_n(p = 1)$:

$$X_n(p = 1) = \int_0^\infty e^{-t} X(t)\, t^n\, dt. \tag{20}$$

Using the same procedure as in Section 1, we get

$$X_n(p = 1) = \lim_{p \to 1} (-1)^n \left[\frac{d^n \bar{X}(p)}{dp^n}\right], \qquad n \geqslant 0. \tag{21}$$

As $X(t)$ is purely oscillatory as $t \to \infty$, the moments $X_n(p = 1)$. $n \geqslant 0$, always exist.

3.2. *Considerations for the Moments X_{-n}, $n > 0$*

The moments X_{-n}, $n > 0$, are a more complex quantity than X_n. We have assumed that $X(t)$ is a bounded function. This is correct for almost any physically measurable parameter. However, $X(t)/t^n$ can be singular at $t = 0$ and therefore the integrals X_{-n} must be tested for convergence at both $t = 0$ and $t \to \infty$. For $t \to \infty$ we apply exactly the same criteria as for X_n, because $X(t)/t^n < X(t)\, t^n$

for $t > 1$. For $t = 0$ we find $X(0)$ by the use of Eq. (10). If $X(0) = 0$ and X_{-n} calculated from (6) does not diverge, we will find the leading term as $t \to 0$ using a series expansion as explained in Section 2.2 (see also (15)). Usually, we need only the first term, so that if the expansion is possible it is also simple. This procedure will assure us of the mathematical existence of X_{-n}.

One important case can occur while $x \neq 0$ (Section 3.1). In this case, we must work with $\bar{X}(p)$ and not $\bar{X}^*(p)$. For a "regular" function, the moment X_{-1} does not converge as $t \to \infty$; therefore we must omit this moment before we start the approximation at $t = 0$.

3.3. *Experimental Considerations*

A block diagram of a typical experiment using the moments method is shown in Fig. 2. The system is driven by a short duration pulse and its response, measured electronically, is analyzed by an on-line computer. The transient recorder is usually needed for storage and as an analog-to-digital converter. The final result is a set of numbers giving the value of the moments. If an on-line computer is not available, the response can be photographed from the oscilloscope or plotted on a regular recorder. (For $n = 0$, the moment is the area under the graph.)

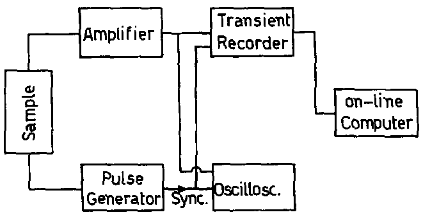

FIG. 2. Block diagram of the electronic setup for a typical experiment using an on-line computer to calculate the moments. The details of the particular experiment are shown schematically as "sample."

The response function $X(t)$ usually has zero amplitude for times much longer than an average time constant of the system ($t \to \infty$). Nevertheless, for greater accuracy in finding X_n, especially for $n > 1$, the contribution of $X(t)$ at large values of t is desired. In this case, the measurement of $X(t)$ is done several times on different time scales and, if needed, a least-squares fit to an exponential function (Ae^{-at}) is used for $t \to \infty$. This exponential fit will be used only as a correction, and only for times t where the fit is better than the experimental error.

For oscillatory systems, the experimental response is multiplied by the function e^{-t} before calculating the pseudomoments, X_n ($p = 1$). Sometimes this function is available in the electronic system, so that the computer program for calculating the pseudomoments does not have to be changed.

4. Examples

4.1. *An LRC Circuit* (Ref [2, Chap. II])

We first analyze the simplest possible system to exhibit the mathematical steps: a series circuit consisting of inductance L, resistance R, and capacity C. The initial current, I_0, and charge, Q_0, are zero. The Laplace transform of the current,

$$\bar{I}(p) = p\bar{E}(p)/(Lp^2 + Rp + C), \tag{22}$$

will be found for three possibilities of external E.M.F.: (1) constant, (2) δ function, (3) a step of length τ.

For a constant E.M.F. $\bar{E}(p) = E/p$, therefore

$$\bar{I}(p) = E/(Lp^2 + Rp + C). \tag{23}$$

This is a "regular" function, and $\lim_{t\to\infty} I(t) = \lim_{p\to 0} p\bar{I}(p) = 0$, so that it has all the moments I_n, $n \geqslant 0$. The response is not oscillatory at $t \to \infty$ from physical considerations, as can also be found experimentally. At $t \to 0$, the limiting value is $\lim_{p\to\infty} pI(p) = 0$. The expansion using (14) is $(E/L)(1/p^2)$, therefore $I(t) \cong (E/L)\, t + 0(t^2)$ at small values of t. Thus, the moments I_{-n}, $n \geqslant 2$, do not exist, I_{-1} is the only existing I_{-n}.

A δ function E.M.F. $E(t) = E\delta(t)$ gives $\bar{E}(p) = E$. Therefore,

$$\bar{I}(p) = pE/(Lp^2 + Rp + C). \tag{24}$$

This function behaves like $\bar{I}(p)$ from (23), but for $t \to 0$, $I(t = 0) = \lim_{p\to\infty} p\bar{I}(p) \neq 0$. Therefore the moments I_{-n}, $n \geqslant 1$, do not exist mathematically.

A step function E.M.F. $E(t) = [1 - H(t - \tau)]$. Therefore,

$$\begin{aligned} \bar{E}(p) &= E[(1/p) - (1/p)\, e^{-p\tau}], \\ \bar{I}(p) &= E(1 - e^{-p\tau})/(Lp^2 + Rp + C). \end{aligned} \tag{25}$$

The function $[E/(Lp^2 + Rp + C)]^{e^{-p\tau}}$ is treated according to (16) and (17). Thus, $\bar{I}(p)$ has all the moments I_n, $n \geqslant -1$, like $\bar{I}(p)$ of (23).

4.2. *Linear Heat Flow in the Solid* $0 < z < l$ *(or Thompson's Cable)*

The general solution of a transmission line of length l shortened at one end and measured at a point a distance z from the open end [2, pp. 269; 3, pp. 110] is

$$\bar{E}(z, p) = \bar{E}(0, p) \sinh q(l - z)/\sinh ql, \qquad 0 < z < l, \tag{26}$$

and

$$\bar{I}(z, p) = \bar{E}(0, p) \cosh q(l - z)/(Lp + R) \sinh ql, \qquad 0 < z < l, \tag{27}$$

where $q^2 = (Lp + R)(Cp + G)$. Here L, C, R, G are the distributed inductance, capacity, resistance, an leakage, respectively. Thompson's cable is defined as a line with $L = G = 0$. In this case, we have the same differential equation as the heat flow equation with $C = \rho C'$, $R = 1/K$, $E = T$, where C' is the heat capacity, ρ is the density, K is the thermal conductivity, T is the temperature, and I is the heat current. We will apply this result to two input conditions: (1) a δ function input of heat $I(0, t) = Q\,\delta(t)$, (2) a constant temperature T applied at $z = 0$. For the first case, from (27), we have for $z = 0$:

$$\bar{I}(0, p) = Q = q\bar{T}(0, p) \cosh ql/(1/K) \sinh ql. \tag{28}$$

Substituting $\bar{T}(0, p)$ from (28) into (26), we get

$$\bar{T}(z, p) = Q \sinh q(l - z)/Kq \cosh ql, \tag{29}$$

where

$$q^2 = (\rho C'/K)p.$$

$\bar{T}(z, p)$ is an even function of q; therefore it is single-valued. It is also bounded in the left half-plane and therefore it is a "regular function." As $t \to \infty$, $\lim_{p\to 0} pT(z, p) = 0$. Therefore all the moments T_n, $n \geqslant 0$, exist. For small values of t, we find $\lim_{p\to\infty} p\bar{T}(z, p) = 0$ $(z \neq 0)$, so that we can expand $\bar{T}(z, p)$ for large p (or q):

$$\begin{aligned} \bar{T}(z, p) &= \frac{Q}{Kq}\,\frac{e^{q(l-z)} - e^{-q(l-z)}}{e^{ql} + e^{-ql}} = \frac{Q}{Kq}\,\frac{e^{-qz} - e^{-q(2l-z)}}{1 + e^{-2ql}} \\ &= \frac{Q}{Kq}\,[e^{-qz} - e^{-q(2l-z)}] \sum_{n=0}^{\infty} (-1)^n\, e^{-2nql}. \end{aligned} \tag{30}$$

The leading term in this expansion for $t \to 0$ is the first one,

$$T(z, t) \approx \frac{Q}{(\pi K t)^{1/2}}\,(e^{-\rho C' z^2/4Kt} - e^{-\rho C'(2l-z)^2/4Kt}), \qquad 0 < z < l. \tag{31}$$

The other terms are smaller, if we do not allow the values $z = 0$ and $z = l$. Using (31), we find that all the moments T_{-n}, $n \geqslant 0$, exist for $0 < z < l$.

We shall use (26) for the case where the input is a constant temperature T at $z = 0$:

$$\bar{T}(z, p) = (T/p)(\sinh q(l - z)/\sinh ql), \tag{32}$$

where $q^2 = (\rho C'/K)p$. This function is "regular" and $\lim_{p\to 0} p\bar{T}(z, p) = T[(l - z)/l]$. Therefore the new function using (19)

$$\bar{T}^*(z, p) = \frac{T}{p}\,\frac{\sinh q(l - z)}{\sinh ql} - \frac{T(l - z)}{pl} \tag{33}$$

has all the moments T_n^*, $n \geqslant 0$. $T(z, 0) = \lim_{p\to\infty} p\bar{T}(z, p) = 0$ and the behavior $t \to 0$ is

$$\bar{T}(z, p) = \frac{T}{p} \frac{e^{q(l-z)} - e^{-q(l-z)}}{e^{ql} - e^{-ql}} = \frac{T}{p} \frac{e^{-qz} - e^{-q(2l-z)}}{1 - e^{-2ql}}$$

$$= \frac{T}{p} [e^{-qz} - e^{q(2l-z)}] \sum_{n=0}^{\infty} e^{-2nql}. \tag{34}$$

The leading term as $t \to 0$ is

$$T(z, t) = T \left\{ \operatorname{erfc} \left[\left(\frac{\rho C'}{Kt} \right)^{1/2} \frac{z}{2} \right] - \operatorname{erfc} \left[\left(\frac{\rho C'}{Kt} \right)^{1/2} \frac{2l - z}{2} \right] \right\}. \tag{35}$$

As $t \to 0$ we use the approximation, $u \to \infty$

$$\operatorname{erfc} u \approx \frac{e^{-u^2}}{\pi^{1/2}} \left(\frac{1}{u} - \frac{1}{2u^3} + \cdots \right). \tag{36}$$

Thus all the moments T_{-n}, $n \geqslant 2$, exist. The moment T_{-1} has to be excluded because of the singularity at $t \to \infty$, as explained in Section 3.2.

To conclude this example, we will show the great advantage of the moments method. We will find T_0 and T_{-1} for the transient case, $I(0, t) = Q\,\delta(t)$.

$$T_0(z) = Q(l - z)/K, \tag{37}$$

$$T_{-1}(z) = \frac{Q}{K} \int_0^\infty \frac{\sinh q(l - z)}{q \cosh ql}\, dp, \tag{38}$$

where $q^2 = (\rho C'/K)p$. Thus

$$T_{-1}(z) = \frac{2Q}{\rho C'} \int_0^\infty \frac{\sinh q(l - z)}{\cosh ql}\, dq = \frac{2Q}{\rho C'} A(z, l), \tag{39}$$

where $A(z, l)$ is the numerical result of the integral, dependent only on z and l. Therefore, an experiment using a thermometer at the point z giving the time dependence of the temperature after a δ-shaped hear pulse was supplied will give us $T_0(z)$ and $T_{-1}(z)$. The external heat pulse $I(0, t) = Q\,\delta(t)$ is known, and one can therefore find K and $\rho C'$, from T_0 and T_{-1} in a single measurement with only one thermometer. A complete treatment of this special case was given in Refs. [6, 7]. The heat capacities of the heater and thermometer, which are always present in such an experiment, were taken in account. The solution of the very complicated time dependent $T(z, t)$ was avoided, using the moments method.

Similar examples can be found in other experiments in physics. The complexity of the experiment itself is reduced and the analysis with an on-line computer enables one to extract physically interesting parameters.

6. Conclusion

The applicability of the moments method had been shown to be very large. All real-life experiments are done on finite-sized geometries. Usually, this fact is regarded as an unwanted complication because of the complex mathematical form of $X(t)$. In our method, the opposite is true. The Laplace transform solution $\bar{X}(p)$ of a finite geometry is usually a "regular" function, so that we are immediately assured that all the moments X_n, $n \geqslant 0$, exist mathematically. In such a case we do not have to make any series expansion or other tests. The method can be applied easily to damped systems with transient inputs using the moments at $p = 0$, and to oscillatory systems by using $p = 1$.

This method has another advantage in an on-line computer experiment. The experiment itself gives a response function, and the computer analyzes this result for the physical parameters of the sample. We do not need a complicated least-squares fit requiring a large memory.

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